Entropy of the symbolic sequence for critical circle maps

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The rabbit sequence, which is related to the golden mean and the Fibonacci numbers, is a self-similar infinite sequence of 0's and 1's that occurs in a variety of contexts in physics and mathematics. For instance, it represents the symbolic dynamics of the nonlinear circle map at the transition from periodicity to chaos, and it appears in mathematical models of quasicrystals. Here, the block entropy for the rabbit sequence is derived analytically. It has already been argued that the sequence exhibits long-range order. Our results confirm this conjecture: The entropy per bit, a direct measure for the long-range order in symbolic sequences, decays logarithmically. The same behavior has been found in a heuristic way for the symbolic dynamics generated by the logistic map at the Feigenbaum point.

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I. INTRODUCTION

In a pioneering work [1], Shannon considered symbolic sequences generated by stationary Markov processes. A symbolic sequence is an infinite string of characters chosen from a finite alphabet. Generalizing Shannon's ideas, McMillan [2] and Khinchin [3] considered stationary and ergodic processes. They established the block entropy as an intuitive and suitable function to measure the "randomness" of a symbolic sequence.

An important measure of randomness is what we now call the block entropy H_n, the uncertainty of a block of length n within the sequence under consideration:

$$H_n := -\sum_i p_i(n) \log_2 p_i(n) \quad , \tag{1}$$

where $p_i(n)$ is the probability to find a block of kind i, if a block of length n from the sequence is randomly chosen. In other words, H_n is the minimum average information (measured in bits) necessary to distinguish one special block of length n from all the others with the same length.

The uncertainty per symbol of a block of length n is $H^{(n)} = H_n/n$. Also important as a measure for the uncertainty (or predictability) of a new symbol after the observation of n symbols emitted by the source is the difference $h_n = H_{n+1} - H_n$, called "entropy per step."

To get a measure which is independent of the block length, i.e., one which describes the source itself, Shannon introduced the uncertainty per step for the infinitely long sequence,

$$h = \lim_{n\to\infty} H^{(n)} \quad , \tag{2}$$

which McMillan named "entropy of the source." Meanwhile, Shannon's theory has been applied to various symbol sources and sequences to investigate their randomness and long-range correlations. Empirically, it has been found that the block entropy scales according to a power law, $H_n \approx an^\mu + b$, $\mu \approx 0.5$, for certain texts [4,5] and $\mu \approx 0.25$ for some classical music [5]. However, other more complicated laws cannot be excluded.

For chaotic and stochastic strings, $h > 0$. For Markov processes of order m, Shannon has shown that the entropy of the source h reaches its limit for $m = n$.

A variety of scaling behaviors have been found for the kneading sequences of nonlinear maps [6]. In fully developed chaos, the scaling is the same as for Markov processes without memory (Bernoulli processes), $H_n \sim n$. If the map generates a dynamic with period p, then $H_{k+p} = \text{const}$, $k > 0$.

The most interesting behavior can be found at the border between periodicity and chaos. The dynamics of the logistic map approaches chaos via period doubling. At the Feigenbaum point, $H^{(n)} \approx \log_2(n)/n$ for large n [7,8], i.e., the entropy per bit decays very slowly; the process has a large memory. The entropy of the source vanishes in this case.

The binary rabbit sequence can be viewed as the symbolic dynamics associated with the critical circle map [9,10], which presents an example for the golden-mean route to chaos. It has already been argued in [10] that the rabbit sequence exhibits long memory tails: Its power spectrum is self-similar with sharp spectral peaks, though the underlying sequence is aperiodic. In the following, an exact formula for the block entropy will be derived. The result confirms the conjecture in [10]: For large n, the entropy per bit is again $H^{(n)} \approx \log_2(n)/n$.

II. DEFINITION OF THE RABBIT SEQUENCE

Let b_r be a sequence of 1's and 0's after r iterations. Let ab be the concatenation of two strings a and b. Then a finite rabbit string can be defined by the following recursion:

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$$\begin{aligned} b_0 &= 0\ , \\ b_1 &= 1\ , \\ b_{r+1} &= b_r b_{r-1}\ . \end{aligned}$$

Thus,

$$\begin{aligned} b_0 &= 0\ , \\ b_1 &= 1\ , \\ b_2 &= 10\ , \\ b_3 &= 101\ , \\ b_4 &= 10110\ , \\ b_5 &= 10110101\ , \quad \text{etc.} \end{aligned}$$

The rabbit sequence is the resulting infinite rabbit string.

III. PROPERTIES AND SIGNIFICANCE OF THE RABBIT SEQUENCE

Many interesting properties of the rabbit sequence and strings can be found in [10] and [11]. First of all, it is obvious that the length of the string b_i equals the $(i+1)$th Fibonacci number F_{i+1}. The Fibonacci numbers can be defined by the recursive relation

$$\begin{aligned} F_0 &= 0\ , \\ F_1 &= 1\ , \\ F_{r+1} &= F_r + F_{r-1}\ . \end{aligned}$$

Therefore the sequence of Fibonacci numbers is $0,1,1,2,3,5,8,13,\ldots$. The Fibonacci numbers and the rabbit strings have a close relation to the golden ratio, $\gamma = \frac{1}{2}(\sqrt{5}-1) = 0.618\ldots$. There is a simple nonrecursive formula for the Fibonacci numbers,

$$F_r = \frac{1}{\sqrt{5}}(\gamma^{-r} - \gamma^r)\ . \tag{3}$$

Thus, for increasing r, the ratio of two successive Fibonacci numbers approaches the golden ratio,

$$\lim_{r\to\infty} F_r/F_{r+1} = \gamma. \tag{4}$$

It is interesting to note that the rabbit sequence is self-similar: The same sequence is obtained if all the "1" bits are replaced by "10," and all "0" bits are replaced by "1". Sometimes this property is used to define the sequence.

Fibonacci introduced the sequence (0,1,1,2,3,5,...) to describe the growth of a rabbit population. The name "rabbit sequence" for the related binary sequence was suggested by Schroeder [11]: After one generation, a young rabbit (0) becomes an old one (1); an old rabbit (1) stays old and generates a young one (10).

Interestingly, the sequence of spins with lowest energy of a one-dimensional antiferromagnetic Ising spin system is the rabbit sequence if a small magnetic field is applied. In this case, "0" stands for "spin down" and "1" for "spin up." Moreover, there are geometrical methods to generate the rabbit sequence which exhibit its close relation to quasicrystals (see [10]).

Finally, the rabbit sequence can also be viewed as the symbolic dynamics of the critical circle maps [9,10] at the transition from periodicity to chaos. A simple example is

$$\theta_{n+1} = \theta_n + \Omega - \frac{1}{2\pi}\sin(2\pi\theta_n)\ .$$

The transition to chaos occurs for $\Omega = \Omega_\infty \approx 0.39$. This parameter corresponds to the Feigenbaum parameter R_∞ for the logistic map. To obtain the rabbit sequence, we set $\theta_0 = 0$ and observe the orbit $\theta_1, \theta_2, \ldots$, and write a "1" if $\theta_i > 0$ and "0" if $\theta_i < 0$.

IV. TWO IMPORTANT LEMMAS

To analyze the entropy of the rabbit sequence, the following two lemmas will be needed.

Lemma A. The strings $b_{r-1}b_r$ and $b_r b_{r-1}$ are identical except in the last two bits, i.e., in position $F_{r+1}-1$ and F_{r+1} (counting starts from 1). The last two bits are either 10 or 01.

Proof. The proof will be given by induction for odd r. The proof for even r is similar.

Obviously, the lemma is correct for $r = 1$.

Now, from the recursion relation $b_{i+1} = b_i b_{i-1}$, it follows that

$$\begin{aligned} b_r b_{r-1} &= b_{r-1}b_{r-2} \quad b_{r-2} \quad b_{r-3} \\ &= b_{r-1}b_{r-2} \quad b_{r-3}b_{r-4} \quad b_{r-3} \\ &= \vdots \\ &= \left(\prod_{i=1}^{r-2} b_{r-i}\right) b_1 b_0 \\ &= \left(\prod_{i=1}^{r-2} b_{r-i}\right) 10 \end{aligned}$$

and

$$\begin{aligned} b_{r-1} b_r &= b_{r-1} \quad b_{r-1} \quad b_{r-2} \\ &= b_{r-1} \quad b_{r-2}b_{r-3} \quad b_{r-3}b_{r-4} \\ &= \vdots \\ &= \left(\prod_{i=1}^{r-2} b_{r-i}\right) b_0 b_1 \\ &= \left(\prod_{i=1}^{r-2} b_{r-i}\right) 01\ . \end{aligned}$$

Comparing $b_r b_{r-1}$ and $b_{r-1}b_r$ completes the proof.

Lemma B. If we discard the first F_r bits of the rabbit sequence, we obtain a sequence which is the same one as before, at least up to position $2F_{r+1}$, with the exceptions at positions F_{r+1} and $F_{r+1}-1$.

Proof. The rabbit sequence begins with the string $b_{r+3} = b_{r+1}b_r b_{r+1} = b_{r-1}b_{r-2}b_{r-1}b_{r-1}b_{r-2}b_{r+1}$. If we discard the first F_r bits, we arrive at a sequence that starts with $b_{r-1}b_{r-1}b_{r-2}b_{r+1}$. This is the same as the first part of the rabbit sequence, with the exceptions at positions $F_{r-1} + F_r - 1 = F_{r+1} - 1$ and F_{r+1} as follows from Lemma A.

The length of the new string is $F_{r-1} + F_{r-1} + F_{r-2} + F_{r+1} = 2F_{r+1}$. This completes the proof for Lemma B.

V. FREQUENCY OF SPECIFIC BLOCKS

In order to calculate the block entropy of the rabbit sequence, it is first necessary to know the frequency of a certain string of length n and kind i.

We will first find the frequency $G_i(n,r)$ of a string of length $n < F_{r-1}$ within a finite initial portion of the rabbit sequence: The first bit of the block i may occur at position 1 up to position F_r of the rabbit sequence. Then we calculate $p_i = \lim_{r\to\infty}[G_i(n,r)/F_r]$.

First consider blocks of length $n < F_{r-1}$ with the first bit of the block at position 1 up to position F_r. Consider the form of the relevant initial part of the rabbit sequence b_{r+1}. Applying Lemma B we know that $F_r - 2$ 0's and 1's within this string repeat after position F_{r-1}. Therefore, an iteration formula is derived as

$$G_i(n,r) = G_i(n,r-1) + G_i(n,r-2) \text{ with } n < F_{r-1} . \tag{5}$$

Indeed, we do not have to be distracted by the fact that the rabbit sequence only repeats until position $P = F_{r+1} - 2$ (Lemma B), since the last bit in the block of maximal length $n_{\max} = F_{r-1} - 1$ is at position $F_r - 1 + n_{\max} = F_{r+1} - 2 = P$.

Now we have to find the possible initial conditions for (5) to get the desired frequencies. For that purpose we have to find the frequencies of blocks of length n with $F_{s-2} \le n < F_{s-1}$.

To understand the following central theorems and their proof, it is helpful to follow the general arguments along the lines of a specific example ($F_s = 13, n = 5,6,7$) which is illustrated in Fig. 1.

Theorems.

(a) For $n = F_{s-2} - 1$, all blocks that start at positions 1 to F_{s-2} are mutually different.

(b) For $n = F_{s-2} - 1$, the blocks that start at positions $F_{s-2} + 1$ to F_{s-1} (i.e., the next F_{s-3} blocks) are repetitions of the first F_{s-3} blocks.

(c) If the block length $n \ge F_{s-2} - 1$ is incremented by one, $n' = n + 1$, the blocks that start at positions 1 to F_{s-2} are still mutually different. The extended block at position $F_s - n'$ becomes different from all blocks at a lower position (marked with "new" in Fig. 1).

(d) This scenario continues as n is incremented until $n = F_{s-1} - 1$. Then all blocks that start at positions 1 to F_{s-1} are mutually different [this is equivalent to Theorem (a) for the next Fibonacci number].

Proof. We will prove the theorems by induction.

Theorem (a) serves as the assumption for the induction. It is easy to check for $n = 1$, $F_s = 5$.

Theorem (b) follows immediately from Lemma B.

To prove (c), consider first the block length $n = F_{s-2}$. This case is illustrated in Fig. 1 as "scenario for $n = 5$". From (a) and (b) it follows that the block of the first $F_{s-2} - 1$ bits of the block at position F_{s-1} has exactly one repetition at a lower position. (In the example, the block is "1101" and the repetition occurs at position 3.) From Lemma B it is clear that the bit at position F_{s-2} of the new block is different. Thus, the new block ("11011") is different from all the blocks at a lower position. If the length of the block is incremented, the new block occurs at one position earlier (see "scenario for $n = 6$"). Therefore the new blocks occur at position $F_{s-1} - n + F_{s-2} = F_s - n$. Hence (c) is true.

Finally, as n reaches $F_{s-1} - 1$, all the blocks from positions 1 to F_{s-1} are mutually different (see "scenario for $n = 7$"). This completes the induction.

It turns out that there are only three different possible initial conditions for (5) defining three classes of blocks, all blocks within one class having the same frequency of occurrence. The possible initial conditions for blocks with $F_{s-1} \le n < F_s$ are

$$\text{(a) } G_a(n,s-1) = 0,\ G_a(n,s) = 1 ,$$

$$\text{(b) } G_b(n,s-1) = 1,\ G_b(n,s) = 1 ,$$

$$\text{(c) } G_c(n,s-1) = 1,\ G_c(n,s) = 2 .$$

[The trivial case $G_i(n,s-1) = G_i(n,s) = 0$ has been omitted.] With (5) we see that the frequencies are again the Fibonacci numbers. Thus the frequencies of different types of blocks up to position F_r are $G_a(n,r) = F_{r-s+1}$, $G_b(n,r) = F_{r-s+2}$, and $G_c(n,r) = F_{r-s+3}$.

From the scenario described above and illustrated in Fig. 1, it is not difficult to derive the number of different blocks for each type of initial condition (a), (b), and (c).

For type (a) we find $a = n - F_{s-1} + 1$.

For type (c) we find $c = F_{s-2} - a = F_s - n - 1$.

For type (b) we find $b = F_{s-1} - c = n - F_{s-2} + 1$.

It is interesting to note that $a + b + c = n + 1$. That is, there are always

block position	Scenario for n=5		Scenario for n=6		Scenario for n=7	
	1 0 1 1 0 1 0 1 1 0 1 1 0		1 0 1 1 0 1 0 1 1 0 1 1 0 1		1 0 1 1 0 1 0 1 1 0 1 1 0 1 0	
1	1 0 1 1 0	new	1 0 1 1 0 1	new	1 0 1 1 0 1 0	new
2	0 1 1 0 1	new	0 1 1 0 1 0	new	0 1 1 0 1 0 1	new
3	1 1 0 1 0	new	1 1 0 1 0 1	new	1 1 0 1 0 1 1	new
4	1 0 1 0 1	new	1 0 1 0 1 1	new	1 0 1 0 1 1 0	new
$5=F_5$	0 1 0 1 1	new	0 1 0 1 1 0	new	0 1 0 1 1 0 1	new
6	1 0 1 1 0	old (same as in position 1)	1 0 1 1 0 1	old (same as in position 1)	1 0 1 1 0 1 1	**new !**
7	0 1 1 0 1	old (same as in position 2)	0 1 1 0 1 1	**new !**	0 1 1 0 1 1 0	new
$8=F_6$	1 1 0 1 1	**new !**	1 1 0 1 1 0	new	1 1 0 1 1 0 1	new
9	1 0 1 1 0	old	1 0 1 1 0 1	old	1 0 1 1 0 1 0	old

FIG. 1. Scenarios for different block lengths (see text).

$$N_n^* = n + 1 \tag{6}$$

different blocks of length n within the rabbit sequence.

VI. BLOCK ENTROPY

The block entropy $\hat{H}_n(r)$ of blocks within the rabbit sequence up to position F_r [from (1)] reads

$$\hat{H}_n(r) = -ap_a \log_2(p_a) - bp_b \log_2(p_b) - cp_c \log_2(p_c)$$

with

$$p_a = \frac{G_a(n,r)}{F_r} \ , \ p_b = \frac{G_b(n,r)}{F_r} \ , \ p_c = \frac{G_c(n,r)}{F_r} \ .$$

By increasing r, we now disclose the probability to find a block at an arbitrary position within the infinite rabbit sequence. Consider

$$p_a = \frac{G_a(n,r)}{F_r} = \frac{F_{r-s+1}}{F_r} = \frac{F_{r-s+1}}{F_{r-s+2}} \frac{F_{r-s+2}}{F_{r-s+3}} \cdots \frac{F_{r-1}}{F_r} \ .$$

For $r \gg 1$, each factor approaches the golden ration γ [see (4)]. Thus, $\lim_{r\to\infty} p_a = \gamma^{s-1}$. Dealing in the same way with blocks of type (b) and (c), we get $p_b \to \gamma^{s-2}$ and $p_c \to \gamma^{s-3}$.

As an example, consider a single bit, i.e., a block of length 1. In this case, $s = 3$. The bit "0" is a block of type (b), i.e., the probability to find a 0 in the rabbit sequence equals γ^2. "1" is a block of kind (a). Thus its probability is γ.

For large r the block entropy of the rabbit sequence becomes

$$H_n = -\Big[a\gamma(s-1) + b(s-2) + c(1+\gamma)(s-3)\Big]\gamma^{s-2}\log_2\gamma \ ,$$

with $F_{s-1} \le n < F_s$. Evaluation of this equation yields

$$H_n = -\Big[(n+1)(2\gamma+1) + (s-2)(\gamma F_{s-2} + F_{s-1}) - \gamma F_{s+1} - F_s\Big]\gamma^{s-2}\log_2\gamma \ . \tag{7}$$

For large block lengths, i.e., for large s, this equation can be simplified. By using

$$F_s = \frac{1}{\sqrt{5}}\gamma^{-s} + O\left(\frac{1}{n}\right) \tag{8}$$

[see (3)] we find after a few steps of derivations for the entropy per bit

$$H^{(n)} = \frac{\log_2 n}{n} + O\left(\frac{1}{n}\right) \ . \tag{9}$$

Exactly the same result has been found in a heuristic way for the kneading sequence of the logistic map [8] (see Fig. 2).

McMillan has shown in [2] that, for stationary and ergodic sources, the average number of different blocks of length n is $N_n^* = 2^{H_n}$ and that the probabilities to find different blocks become equal for all possible blocks that might occur. For the rabbit sequence, the latter is not true; the probabilities for the three different classes of blocks (a), (b), and (c) differ for large block lengths by a factor γ, cf. γ^2. This is because the rabbit sequence is not stationary. However, from (6) and (9) we see that $N_n^* = 2^{H_n}$ is still true for large n.

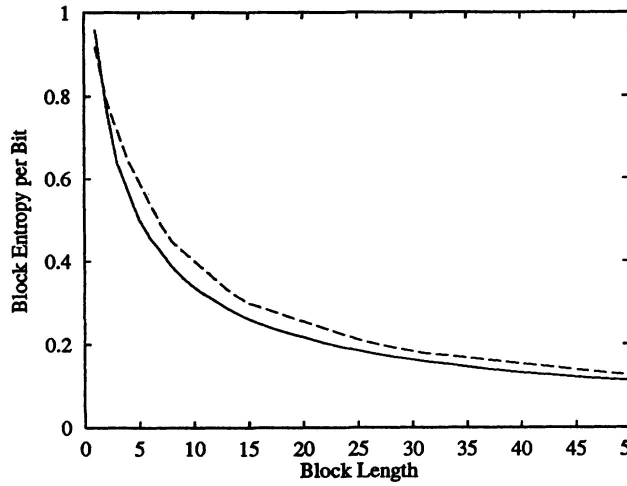

FIG. 2. Block entropy per bit $H^{(n)}$ for the symbolic dynamics of the circle map (solid line) and of the logistic map (dashed line, after [8]).

Finally, from (7), we find a simple law for the predictability of a new bit after the observation of n bits, $h_n = H_{n+1} - H_n$:

$$h_n = -\sqrt{5}\,\gamma^{s-2}\log_2\gamma \ , \quad F_{s-1} \le n < F_s \ .$$

For large s, this yields the power law $h_n \sim 1/n$. Again, this is the same scaling as for the dynamics of the logistic map. The entropy of the source vanishes in both cases.

VII. CONCLUSION

Exact expressions for the block entropy and entropy per step for the rabbit sequence have been derived. For $n \gg 1$, the entropy per bit $H^{(n)}$ scales exactly as the entropy of the symbolic dynamics generated by the logistic map at the Feigenbaum point. In both cases, long memory tails can be observed. This confirms suspicions that long-range order is typical for the dynamics of systems at the transition from order to chaos. Moreover, there is evidence that the entropies of critical dynamics obey similar scaling laws.

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